Radio Interferometers
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Why Interferometry?

- Resolution and collecting area

Telescope size, surface accuracy, and pointing are jointly limited

\[ \theta_{res} \approx 2'' \times \frac{\lambda_{cm}}{D_{km}} \]
Limited?

NRAO 300ft Telescope
Why Interferometry?

- Resolution and collecting area

  Telescope size, surface accuracy, and pointing are jointly limited

- Interferometers can provide:
  - highest resolution (EHT: $\frac{\lambda}{D} > 10G\lambda$!)
  - largest collecting area
  - large number of resolution elements
    large field of view with high sensitivity
  - highest astrometric precision
Antenna Illumination and Beam

Consider a 1-D antenna of length $D$ transmitting at frequency $\nu$ ($\lambda = c/\nu$).

Calculate field at point $A$ at large distance $R$.

Consider small segment $dx$ at position $x$, with field $g(x)$.

Electric field contribution at point $A$ is

$$dE \propto \frac{g(x)}{r(x)} \exp(-i2\pi r(x)/\lambda) \, dx$$
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In coefficient: $1/r(x) \simeq 1/R$

In exponent: $r(x) = R + xsin\theta \approx R + xl$ (for small $\theta$, and $l \equiv sin\theta$)

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Now, define $u = x/\lambda$, absorb constants into $g$, and integrate to get $E(A)$

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$$E = \int g(u) e^{-i2\pi ul} \, du$$

- Antenna beam is **Fourier Transform** of antenna illumination pattern $g(x)$!
Antenna Illumination and Beam

- Larger aperture in $\lambda/D \rightarrow$ smaller beam on sky (FT similarity theorem)

- Actual antennas usually “taper” $g(u)$

- Blockage, surface errors, etc., can be included in beam pattern via FT
Antenna Illumination and Beam

- Two small apertures -> plane wave

Antenna Illumination Pattern

Field Pattern

Power Pattern (PSF)
Antenna Illumination and Beam

• Two small apertures -> plane wave
• Wider separation -> higher spatial frequency

Antenna Illumination Pattern

Field Pattern

Power Pattern (PSF)
Two-Element Analysis

Assumptions:
- Distant point source (→ plane wave)
- Monochromatic

\[
V_1 = V \cos 2\pi \nu (t - \tau_g) \\
\tau_g = \frac{D \sin \theta}{c}
\]
Two-Element Analysis

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• Distant point source (→ plane wave)
• Monochromatic

\[
\tau_g = \frac{D \sin \theta}{c}
\]

\[
V_1 = V \cos 2\pi\nu(t - \tau_g)
\]

\[
F = \langle V_1 V_2 \rangle = \langle V^2 \cos \omega t \cos \omega(t - \tau_g) \rangle
\]

\[
= \frac{V^2}{2} \langle \cos \omega \tau_g + \cos(2\omega t - \omega \tau_g) \rangle
\]

0 when averaging over \( t \gg 1/(2\omega) \)
Two-Element Analysis

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- Distant point source (→ plane wave)
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\[ \tau_g = \frac{D \sin \theta}{c} \]

\[ F = \langle V_1 V_2 \rangle = \langle V^2 \cos \omega t \cos \omega (t - \tau_g) \rangle \]
\[ = \frac{V^2}{2} \cos \omega \tau_g \]
\[ F = \frac{V^2}{2} \cos \left( 2\pi \frac{D}{\lambda} \sin \theta \right) \]

Periodic oscillation, as \( D \sin \Theta \) changes by \( \lambda \)

TMS 2.1, 2.2
Two-Element Analysis

Monochromatic 1GHz signal

Antenna 1

Antenna 2

$F = 0.5$

A/B in phase

$F = 0$

A/B in quadrature phase ($D \sin \Theta = 1, 3, 5, \ldots \lambda/4$)

$F = -0.5$

A/B in anti-phase

Fig: Anna Scaife
Two-Element Analysis

\[ V_1 = V \cos 2\pi \nu (t - \tau_g) \]
\[ \tau_g = \frac{D \sin \theta}{c} \]

\[ F = \langle V_1 V_2 \rangle = \langle V^2 \cos \omega t \cos \omega (t - \tau_g) \rangle \]
\[ = \frac{V^2}{2} \cos \omega \tau_g \]
\[ F = \frac{V^2}{2} \cos \left( \frac{2\pi D}{\lambda} \sin \theta \right) \]

"Correlator": Multiply and Average

Complex correlator: 
F’ provides missing fringe information
Two-Element Analysis

Resolution?

\[ \Delta \left( 2\pi \frac{D}{\lambda} \sin \theta \right) \sim 2\pi \]

\[ \frac{D}{\lambda} \Delta (\sin \theta) \sim 1 \]

\[ \Delta \theta \sim \frac{\lambda}{D} \]

“Fringe spacing” is determined by telescope separation
Two-Element Analysis

What if signal is not monochromatic?

Assume bandwidth $\Delta \nu$ around $\nu_0$

$$F = \frac{1}{\Delta \nu} \int_{\nu_0-\Delta \nu/2}^{\nu_0+\Delta \nu/2} \frac{V^2}{2} \cos \left[ \frac{2\pi D\nu}{c} \sin \theta \right] d\nu$$

$$F = \frac{V^2}{2} \cos \left[ \frac{2\pi D\nu_0}{c} \sin \theta \right] \sin \left[ \frac{\pi D\Delta \nu}{c} \sin \theta \right] \sin \left[ \frac{\pi D\Delta \nu}{c} \sin \theta \right]$$

$$= F_{\nu_0} \text{sinc} \left[ \frac{D\Delta \nu}{c} \sin \theta \right] = F_{\nu_0} \text{sinc} \left[ \tau_g \Delta \nu \right]$$

Same fringe, with sinc ($=\sin(\pi x)/(\pi x)$) envelope

$\tau_g = \frac{D \sin \theta}{c}$
Two-Element Analysis

What if signal is not monochromatic?

Assume bandwidth $\Delta \nu$ around $\nu_0$

$$F = F_{\nu_0} \text{sinc}[\tau_g \Delta \nu]$$

Bandwidth restriction is extremely stringent for useful baseline lengths

$$\Delta \nu \ll 1/\tau_g$$

1km baseline implies $\Delta \nu \ll 300\text{kHz}$

Interferometers designed for imaging small scales must remove $\tau_g$

This is “Delay tracking”
Interferometric Measurement

• Idealized interferometer
  • Two antennas, \( A_1 \) at \((x,y)\) and \( A_2 \) at \((0,0)\) for simplicity
  • Source in direction \( r \) emitting electric field \( \mathcal{E}(x',y') \)

• Simplifying assumptions (can be relaxed)
  • Source is very distant
  • Monochromatic source
  • Ignore polarization
  • Source is spatially incoherent
  • Nothing between antennas and source
Interferometric Measurement

• What do we measure when we multiply E fields at two antennas?

Field received at Antenna 1 (similar for A_2):

$$E_1(x', y', t) = \frac{\mathcal{E}(x', y', t - r_1/c)}{z} e^{-2\pi i \nu (t - r_1/c)}$$

Define the spatial correlation function between fields measured at positions of A_1 and A_2

$$R_{12}(x', y', r_1, r_2) = \langle E_1 E_2^* \rangle$$

$$= \frac{1}{z^2} \langle \mathcal{E}(x', y', t) \mathcal{E}^*(x', y', t) \rangle e^{2\pi i \nu (r_1 - r_2)/c} dx' dy'$$

• Note: this is time-averaged source intensity I(x', y')

$$\langle \mathcal{E}(x', y', t) \mathcal{E}^*(x', y', t) \rangle \equiv I(x', y')$$

* Assumed sampled bandwidth small compared to frequency so that field is similar across propagation time difference to drop r from $\mathcal{E}$
Interferometric Measurement

• What do we measure when we multiply E fields at two antennas?

• Simplify distance difference:
  • If source far away, \( z \gg (x'-x) \), and so:
    \[
    r_1 = \sqrt{z^2 + (x' - x_1)^2 + (y' - y_1)^2} \approx z \left[ 1 + \frac{(x' - x_1)^2}{2z^2} + \frac{(y' - y_1)^2}{2z^2} \right]
    \]
    \[
    r_2 = \sqrt{z^2 + x'^2 + y'^2} \approx z \left[ 1 + \frac{x'^2}{2z^2} + \frac{y'^2}{2z^2} \right]
    \]
  • Difference between \( r_1 \) and \( r_2 \):
    \[
    r_1 - r_2 \approx -\frac{1}{z} \left[ x'D_x + y'D_y - \frac{1}{2} (D_x^2 + D_y^2) \right]
    \]
    \( (D_x = x_1 - x_2) \)
  • And if wavefront curvature small (in “far field”), then:
    \[
    r_1 - r_2 \approx -\frac{x'D_x}{z} - \frac{y'D_y}{z}
    \]
Interferometric Measurement

- What do we measure when we multiply E fields at two antennas?

- Define convenient coordinates
  - Convert $x'$, $y'$ to angles:
    \[
    l = \frac{x'}{z}, \quad m = \frac{y'}{z}, \quad \frac{dx'}{z} = dl, \quad \frac{dy'}{z} = dm
    \]
  - Express $D_x$, $D_y$ in wavelengths
    \[
    u = \frac{D_x}{\lambda}, \quad v = \frac{D_y}{\lambda}
    \]
  - Simplify:
    \[
    \frac{\nu}{c} (r_1 - r_2) = \frac{r_1 - r_2}{\lambda} \approx -(ul + vm)
    \]
**Interferometric Measurement**

- What do we measure when we multiply E fields at two antennas?

- Put it all back together:

\[
R_{12}(x', y', r_1, r_2) = \langle E_1 E_2^* \rangle
= \frac{1}{z^2} \langle \mathcal{E}(x', y', t) \mathcal{E}^*(x', y', t) \rangle e^{2\pi i u (r_1 - r_2)/c} dx' dy'
\]

\[
I(x', y')
\]

\[
l = \frac{x'}{z} \quad m = \frac{y'}{z}
\]

\[
\frac{dx'}{z} = dl \quad \frac{dy'}{z} = dm
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Interferometric Measurement

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\[ = \frac{1}{z^2} \langle \mathcal{E}(x', y', t) \mathcal{E}^*(x', y', t) \rangle e^{2\pi i \nu (r_1 - r_2)/c} \, dx' \, dy' \]

- This becomes:

\[ R_{12}(x', y') = I(l, m) e^{-2\pi i (ul + vm)} \, dl \, dm \]
Interferometric Measurement

- What do we measure when we multiply E fields at two antennas?

- Put it all back together:

\[ R_{12}(x', y', r_1^2, r_2^2) = \langle E_1 E_2^* \rangle \]

\[ = \frac{1}{z^2} \langle \mathcal{E}(x', y', t) \mathcal{E}^*(x', y', t) \rangle e^{2\pi i \frac{(r_1 - r_2)}{c} dx' dy'} \]

- This becomes:

\[ R_{12}(x', y') = I(l, m) e^{-2\pi i (ul + vm)} dl \, dm \]

- Now, define “visibility” as integral of \( R_{12} \) over sky:

\[ V(u, v) = \iint I(l, m) e^{-2\pi i (ul + vm)} dl \, dm \]

It's a Fourier Transform!

One spatial frequency \((u,v)\) measured per baseline
Fourier Sampling

• Of course, interferometers only sample some $uv$ spacings

Example: ALMA

Array configuration

Instantaneous sampling

With Earth rotation
Fourier Sampling

- Of course, interferometers only sample some $uv$ spacings
  
  - Largest $uv$ distances determine resolution
  - Inner $uv$ hole $\rightarrow$ Missing large-scale emission (sky is high-pass filtered)
Fourier Sampling

- Baseline coordinates \((u, v, w)\) are projection of telescope separation into plane perpendicular to source direction \((u, v)\) and toward source \((w)\)
- Baseline coordinates: X, Y, Z
  - For VLBI, X Y defined using Greenwich meridian
Fourier Sampling

- Baseline coordinates \((u, v, w)\) are projection of telescope separation into plane perpendicular to source direction \((u, v)\) and toward source \((w)\)

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix} =
\begin{bmatrix}
  \sin H & \cos H & 0 \\
  -\sin \delta \cos H & \sin \delta \sin H & \cos \delta \\
  \cos \delta \cos H & -\cos \delta \sin H & \sin \delta
\end{bmatrix}
\begin{bmatrix}
  X_\lambda \\
  Y_\lambda \\
  Z_\lambda
\end{bmatrix}.
\]

\(H =\) Hour Angle \\
\(\delta =\) Declination
Fourier Sampling

- Baseline coordinates \((u, v, w)\) are projection of telescope separation into plane perpendicular to source direction \((u, v)\) and toward source \((w)\)

\[
\begin{bmatrix}
u \\
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\cos \delta \cos H & -\cos \delta \sin H & \sin \delta
\end{bmatrix}
\begin{bmatrix}
X_\lambda \\
Y_\lambda \\
Z_\lambda
\end{bmatrix}
\]

- From above, \(u, v\) can be rearranged:

\[
u^2 + \left(\frac{v - Z_\lambda \cos \delta_0}{\sin \delta_0}\right)^2 = X_\lambda^2 + Y_\lambda^2.
\]

- Offset in \(v\) : \(Z \cos \delta\)
- Ellipsoid axis ratio: \(\sin \delta\)
Image Synthesis

- Fourier transform of the uv coverage is your “dirty beam”
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• Fourier transform of the uv coverage is your “dirty beam”
Fourier Relations in Interferometry

• Some sample visibility functions

• Reminder: Visibility function is complex number

• Complex correlator provides in-phase and quadrature-phase outputs, giving the real (cosine) and imaginary (sine) components

• Can also be expressed as amplitude and phase

\[ V(u, v) = V_R + i V_I = |V| e^{i\theta} \]
100 Jansky point source offset from phase center
Visibility amplitude for 100 Jy point source
Visibility phase for offset 100 Jy point source spacing between cycles is inversely prop. to distance of source from phase center
100 Jy gaussian source offset from phase center
Visibility amplitude for 100 Jy gaussian source

Visibility amplitude at (U,V)=(0,0) is total flux in source
Visibility phase for offset 100 Jy gaussian source

Deviations from straight lines are due to computational rounding where visibility amplitude is very low.
Visibility phase reveals the location of the emission!

Deviations from straight lines are due to computational rounding where visibility amplitude is very low.
\[ |\text{Shep}| \exp(i \phi_{\text{Roomba}}) \]

\[ |\text{Roomba}| \exp(i \phi_{\text{Shep}}) \]
|Shep| * exp(i* $\phi_{\text{Roomba}}$)

|Roomba| * exp(i* $\phi_{\text{Shep}}$)
100 Jy elliptical unif. disk and 50 Jy circular unif. disk
Visibility amplitude for elliptical plus circular disks

Visibility amplitude at $(U,V)=(0,0)$ is total flux from both sources, 150 Jy.

Sharp edges in source create lots of high spatial frequency content (note log stretch in color wedge)
Visibility amplitude for elliptical plus circular disks

Visibility amplitude at (U,V)=(0,0) is total flux from both sources, 150 Jy.

Sharp edges in source create lots of high spatial frequency content (note log stretch in color wedge)

Emission structures add as complex numbers in uv plane
Fourier Relations in Interferometry

- Fourier transform of the uv coverage is your “dirty beam”
Fourier Relations in Interferometry

- uv coverage (FT of beam) **multiplies** 2D visibility function (FT of image)
  
  So “dirty beam” is **convolved** with true image
Fourier Relations in Interferometry

- uv coverage (FT of beam) **multiplies** 2D visibility function (FT of image)
  So beam is **convolved** with true image
- Deconvolution techniques are central to understanding these “images”
Closing Remarks

• Fill out the evaluation!  
  bit.ly/BH_Interferometry_Survey

• Future webinars!
  • March 11: VLBI Data Series: Session 1 - Handling Data, Managing Errors
  • March 18: VLBI Data Series: Session 2 - Imaging Techniques
  • May 5: VLBI Data Series Session 3 - Model Comparison